Metric Spaces and Topology
Lecture 22

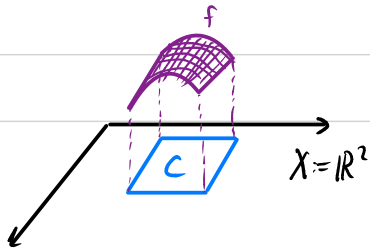
By rescaling, Urysohn's lemma can be stated as follows:

Urysohn's Lemma. If X is a normal top space, $d \geq 0$, and $C_0, C_1 \subseteq X$ disjoint closed sets, then there is a continuous function $f: X \to [0,d]$ s.t. $f|_{C_0} \equiv 0$ and $f|_{C_1} \equiv d$.

Note that Urysohn's lemma can be viewed as a continuous extension statement: give the function $g: C_0 \sqcup C_1 \to [0,1]$ s.t. $g|_{C_0} \equiv 0$, $g|_{C_1} \equiv 1$ (in particular, g is continuous on $C_0 \sqcup C_1$ because preimages of closed are closed), it admits a continuous extension $\bar{g}: X \to [0,1]$.
Iterative applications of this gives the more general statement:

Tietze Extension theorem. Let X be a normal top space, and $f: C \to [0,1]$ be a continuous function on a closed subset $C \subseteq X$. Then f admits a continuous extension $\bar{f}: X \to [0,1]$.

Proof. We will build a sequence $(f_n)_{n=1}^{\infty}$ of



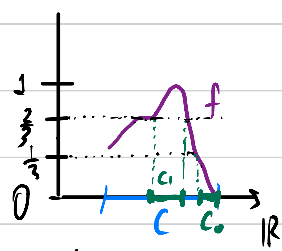
continuous functions $f_n : X \to [0, \frac{2^{n-1}}{3^n}]$ s.t. $0 \leq f - \sum_{i=1}^{n} f_n|_C \leq \frac{2^n}{3^n}$

Given such a sequence, note that:

(i) $\|f_n\|_u := d_u(f_n, 0) \leq \frac{2^{n-1}}{3^n}$, hence the tail $\sum_{n \geq N} f_n$ converges to the constant 0 function in the uniform metric by the triangle inequality: $\|\sum_{n=N} f_n\|_u \leq \sum_{n \geq N} \|f_n\|_u \leq \sum_{n \geq N} \frac{2^{n-1}}{3^n} \xrightarrow[N \to \infty]{} 0$.

Thus, $\left(\sum_{n=1}^{N} f_n\right)_N$ is Cauchy in the uniform metric, hence it has a limit $\bar{f}$ in the uniform metric, which is thus continuous since $C(X, \mathbb{R})$ is a complete metric space with the uniform metric. Furthermore, $0 \leq \bar{f} \leq 1$ because $\forall N$, $\sum_{n=1}^{N} f_n \leq \sum_{n=1}^{N} \frac{2^{n-1}}{3^n} \leq 1$.

(ii) $\bar{f}|_C = f$ because $d_u(\bar{f}|_C, f) = \lim_n d_u\left(\sum_{i=1}^{n} f_i|_C, f\right) \leq \lim_n \frac{2^n}{3^n} = 0$ on C.



Now we build such a sequence recursively. Let $C_0 := f^{-1}([0, \frac{1}{3}])$ and $C_1 := f^{-1}([\frac{2}{3}, 1])$, so these are disjoint and closed by continuity. By Urysohn, $\exists$ continuous $f_1 : X \to [0, \frac{1}{3}]$ s.t. $f_1|_{C_0} \equiv 0$ and $f_1|_{C_1} \equiv \frac{1}{3}$. Then $f - f_1|_C \geq 0$ and $f - f_1|_C \leq \frac{2}{3}$. Then we define

$f_2 : X \to [0, \frac{2}{3^2}]$ similarly using $f - f_1|_C$ instead of f.
Continuing this way, we get $f_n : X \to [0, \frac{2^{n-1}}{3^n}]$ s.t.
$0 \le f - \sum_{i=1}^{n} f_i|_C \le 2^n/3^n$. $\square$

Tietze Extension for unbdd functions. Let X be a normal top space and let $f : C \to \mathbb{R}$ be a continuous function on a closed $C \subseteq X$. Then f admits a continuous extension $\bar{f} : C \to \mathbb{R}$.

Proof. Let $g := f/(1+|f|)$. Then $-1 < g < 1$ and $g : X \to (-1, 1)$ is continuous. We can apply the bounded Tietze extension theorem to g (actually, to $\frac{1}{2}(g+1)$) and obtain a continuous extension $\bar{g} : X \to [-1, 1]$, i.e. $\bar{g}|_C = g$.
To go back to f, we would want to take
$\bar{f} := \bar{g}/1-|\bar{g}|$ but $\bar{g}$ might take ± 1 values.

Solution (Vahagh, Erik). Treat $\frac{\bar{g}}{1-|\bar{g}|}$ as a continuous function from X to $\overline{\mathbb{R}} := \mathbb{R} \cup \{\pm \infty\}$ and take $\bar{f} := \max(\min(1, \bar{g}/1-|\bar{g}|), -1)$. This is cont. on X and has values in $\mathbb{R}$.
Furthermore, $\bar{f}|_C = f$.

Solution 2. Let $C_0 := \bar{g}^{-1}(\{\pm 1\})$ and $C_1 := C$, then C_0, C_1 are

disjoint closed sets, so by Urysohn, $\exists$ cont.
$h : X \to [0,1]$ s.t. $h|_{C_0} \equiv 0$ and $h|_{C_1} \equiv 1$,
and we take $\bar{f} := (\bar{g} \cdot h) / 1 - |\bar{g} \cdot h|$.
Then $\bar{f} : X \to \mathbb{R}$ w/ $f|_C = f$.

Remark. This shows that Tietze extension holds respecting open interval ranges. $\square$

Compactness. Compactness is the analogue of finite in the continuous world.

Def (open covers). A top. space X is called *compact* if every open cover admits a finite subcover.

$\Updownarrow$

Def (intersection of closed). A top. space X is called *compact* if every family $\mathcal{F} \subseteq P(X)$ of closed sets with the *finite intersection property* (i.e. every finite subcollection $F_1, F_2, \dots, F_n \in \mathcal{F}$ has a nonempty intersection: $\bigcap_{i=1}^{n} F_i \neq \emptyset$) has an intersection, i.e. $\underbrace{\bigcap \mathcal{F}}_{\bigcap_{F \in \mathcal{F}} F} \neq \emptyset$.

Equivalence of the two defs. $\mathcal{U} \subseteq \mathcal{P}(X)$ is an open cover

$\Leftrightarrow \mathcal{F}_\mathcal{U} := \{U^c : U \in \mathcal{U}\}$ is such that $\bigcap \mathcal{F}_\mathcal{U} = \emptyset$.

Conversely, if $\mathcal{F}$ is a family of closed sets, then $\bigcap \mathcal{F} = \emptyset \Leftrightarrow \mathcal{U}_\mathcal{F} := \{F^c : F \in \mathcal{F}\}$ is an open cover.

open covers $\Rightarrow$ inters. of closed. Let $\mathcal{F}$ be a family with the fin. int. property. Then if $\bigcap \mathcal{F}$ were $\emptyset$, $\mathcal{U}_\mathcal{F}$ would be an open cover with no finite subcover.

inters. of closed $\Rightarrow$ open covers. Let $\mathcal{U}$ be an open cover, so $\bigcap \mathcal{F}_\mathcal{U} = \emptyset$ thus $\mathcal{F}_\mathcal{U}$ cannot have the fin. int. prop., hence $\mathcal{U}$ cannot have a finite subcover.

$\square$